PRIME CHAINS

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ABSTRACT. A number that is divisible by a given prime can be added (or subtracted) to a non-zero number which is indivisible by the same prime to create an integer which is indivisible by that prime. This can be used for a given set of primes to produce a new integer, N, which is indivisible by every prime in the set. And, if the set of primes contains every prime less than a given prime, p_n, and if $1 < N < p_n^2$, then N is prime.

Theorem 1. *For any prime, p_a, the sum (or difference) of a number which is divisible by the prime and a non-zero number which is indivisible by the prime, is an integer which is indivisible by the prime, p_a.*

Proof. Let p_a be a prime, let F and G be non-zero numbers and let G be indivisible by p_a. Then, $(p_a^i * F) \pm G = N$ for some $i \geq 1$ and some integer, N. Clearly, $p_a^i * F$ is divisible by p_a since p_a is a factor of the composite. Assume for the sake of contradiction, that N is divisible by p_a, then $N = p_a^j * H$ for some $j \geq 1$ and some non-zero integer, H. Hence, $(p_a^i * F) \pm G = (p_a^j * H)$. So, $\pm G = (p_a^j * H) - (p_a^i * F) = p_a((p_a^{j-1} * H) - (p_a^{i-1} * F))$. Hence, p_a is a factor of G, which is a contradiction, since G was defined to be indivisible by p_a. Therefore, the N is indivisible by p_a. □

Theorem 2. *Full Prime Chain*
For the set of all unique primes less than p_n, the number, N, is prime, if $1 < N < p_n^2$ and $\pm(p_1^0 * p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n}) \pm (p_1^{j_1} * p_2^0 * p_3^{j_3} * \ldots * p_n^{j_n}) \pm \ldots \pm (p_1^{k_1} * p_2^{k_2} * p_3^{k_3} * \ldots * p_n^0) = N$ for some numbers $i_2, i_3, \ldots, i_n, j_1, j_3, \ldots, j_n, \ldots, k_1, k_2, k_3, \ldots, k_{n-1} \geq 1$, where the set of primes is represented as $(p_1, p_2, p_3, \ldots, p_n)$ and n is the number of primes in the set.

Lemma 1. *For any set of unique prime numbers, $(p_1, p_2, p_3, \ldots, p_n)$, the number, N is indivisible by any prime in the set when, $\pm(p_1^0 * p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n}) \pm (p_1^{j_1} * p_2^0 * p_3^{j_3} * \ldots * p_n^{j_n}) \pm \ldots \pm (p_1^{k_1} * p_2^{k_2} * p_3^{k_3} * \ldots * p_n^0) = N$ for any $i_2, i_3, \ldots, i_n, j_1, j_3, \ldots, j_n, \ldots, k_1, k_2, k_3, \ldots, k_{n-1} \geq 1$.*

Proof. Lemma 1
Assume for the sake of contradiction, that there exists a prime, p_a, in the set that divides N.
Then, by the way it was constructed, the summation can be written, $\pm(p_a^0 * p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n}) \pm (p_a^{j_1} * p_2^0 * p_3^{j_3} * \ldots * p_n^{j_n}) \pm \ldots \pm (p_a^{k_1} * p_2^{k_2} * p_3^{k_3} * \ldots * p_n^0) = N$.
Since, $i_2, i_3, \ldots, i_n, j_1, j_3, \ldots, j_n, \ldots, k_1, k_2, k_3, \ldots, k_{n-1} \geq 1$, the formula can be re-written, $\pm(p_a^0 * p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n}) \pm p_a((p_a^{j_1-1} * p_2^0 * p_3^{j_3} * \ldots * p_n^{j_n-1}) \pm \ldots \pm (p_a^{k_1-1} * p_2^{k_2} * p_3^{k_3} * \ldots * p_n^0)) = N$.
And, since N was assumed to be divisible by p_a, then $N = p_a^x M$ for some integers

Date: Dec 27th, 2020.
Key words and phrases. Number Theory, Prime Numbers, Prime Chain.

$x \geq 1$ and $M \neq 0$.
Additionally, $p_a^0 = 1$, so $\pm(p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n}) \pm p_a((p_a^{j_1-1} * p_2^0 * p_3^{j_3} * \ldots * p_n^{j_n-1}) \pm \ldots \pm (p_a^{k_1-1} * p_2^{k_2} * p_3^{k_3} * \ldots * p_n^0)) = p_a^x M$.
And, $p_a(\pm(p_a^{j_1-1} * p_2^0 * p_3^{j_3} * \ldots * p_n^{j_n-1}) \pm \ldots \pm (p_a^{k_1-1} * p_2^{k_2} * p_3^{k_3} * \ldots * p_n^0)) - p_a^x M = \mp(p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n})$.
Hence, $p_a(\pm(p_a^{j_1-1} * p_2^0 * p_3^{j_3} * \ldots * p_n^{j_n-1}) \pm \ldots \pm (p_a^{k_1-1} * p_2^{k_2} * p_3^{k_3} * \ldots * p_n^0) - p_a^{x-1} M) = \mp(p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n})$.
Thus, $\mp(p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n})$ must be divisible by p_a.
However, p_a does not divide $\mp(p_2^{i_2} * p_3^{i_3} * \ldots * p_n^{i_n})$, since the Fundamental Theorem of Arithmetic states that a value is uniquely represented by a product of primes and p_a is not in that product, which is a contradiction.
Therefore, N is not divisible by the primes in the set. □

Proof. Full Prime Chain
Assume for the sake of contradiction that N is not prime.
Then, N is a composite. And, since N is a composite, then N has a prime factor not exceeding $\sqrt{N}$. [1]
Since, $N < p_n^2$, then N has a prime factor less than p_n.
However, by the lemma, N is indivisible by any prime in the set of primes, which means N does not have a prime factor less than p_n, which is a contradiction.
Therefore, N is prime. □

Prime Chains

It could be said that the prime chain defined above was a full (or complete) prime chain, since it used the maximum number of products to create the chain. A smaller number of products can be used to create a prime chain by grouping together primes to the zeroth power. Or, stated more precisely, each prime in the prime chain must be missing in exactly one of the products to produce a value which is indivisible by all the primes in the set, however, it does not matter which product the prime is missing from (as long as the summation has two or more products). For example, for the prime set, $(2, 3, 5)$, the prime chain could be $(3 * 5) + 2 = 17$ or $(2 * 3) + 5 = 11$, and a full prime chain could be $(2 * 3) - (2 * 5) + (3 * 5) = 11$. The same proofs used above can be used for partial chains with the only difference being the number of products that are grouped with the chosen prime, p_a.

References

[R] Kenneth H. Rosen, (2005). *Elementary Number Theory and Its Applications, Fifth Edition* Pearson Addison Wesely, Boston.

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[1] Kenneth H. Rosen, (2005). *Elementary Number Theory and Its Applications, Fifth Edition* Pearson Addison Wesely, Boston, p69.