CARDINALITY AND THE CONTINUUM HYPOTHESIS

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Dedicated to my father, Walter, and to every other scientist who spent their life on the edge of an epiphany without finding it.

Abstract. This paper seeks to dispel the notion that the natural numbers and the real numbers have different cardinalities. In order to do so, the proof to Cantor's Theorem and the proof that $(0, 1) \not\approx \mathbb{N}$ will be disproven. And, in the process, it will be shown that the natural numbers cannot be treated as both a set and a sequence simultaneously. Finally, the natural numbers will be shown to be equivalent to the real numbers and the power set of the natural numbers.

Introduction

Cardinality is commonly believed to differentiate between the size of the natural numbers and the real numbers. In truth, however, cardinality shows the two sets to be equivalent. This misconception arises from two flawed proofs, namely the proof for Cantor's Theorem and the proof that $(0, 1) \not\approx \mathbb{N}$. In this paper, the flaw in each proof will be demonstrated and discussed at length. Cantor's Theorem will be addressed first, since it provides the basis for increasing levels of cardinality. Then, there will be a discussion of the error in the proof that $(0, 1) \not\approx \mathbb{N}$, which centers around the lack of distinction between the natural numbers as a set (ZFC Axioms) and the natural numbers as a sequence (Peano's Axioms). Finally, it will be shown that the natural numbers (as both a set and a sequence) are equivalent to the real numbers and the power set of the natural numbers.

Disproving the proof to Cantor's Theorem

The flaw in the proof to Cantor's Theorem stems from an assumed relationship between the elements of the domain, A, and the co-domain, $\mathcal{P}(\mathcal{A})$. Specifically, the proof assumes that every element in the domain, A, must be directly related to the same element in the co-domain, $\mathcal{P}(A)$. This restricts the function, g, to a subset of the possible mappings from $\mathcal{A}$ to $\mathcal{P}(\mathcal{A})$. Which means, the arbitrary function in the proof is not actually arbitrary. [1]

The mapping is obscured by the fact that both sets are composed of the same underlying symbols. This is most evident in the variable, B, which is constructed to create a contradiction. Due to the lack of distinction between the domain and

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[1] "Assume that there exists a bijection, g, then show a contradiction" is logically equivalent to the statement that "For any function, g, assume that g is a bijection, then show a contradiction."

co-domain, the variable is allowed to be both a member of the co-domain and a subset of the domain simultaneously. In order to remove the ambiguity between the domain and the co-domain, the proof will be re-written with two equivalent sets that do not share the same symbols.

Since the set, A, is any generic set, we can choose A to be the set of odd natural numbers. Then, the power set of A becomes the power set of the odd natural numbers. And, since the power set of the even natural numbers is equivalent to the power set of the odd natural numbers, the proof should still hold when the power set of the odd natural numbers is replaced with the power set of the even natural numbers. However, applying (the second part of) Cantor's proof to these sets yields the following:

Disproof: Cantor's Theorem

Let $\mathcal{O}$ be the odd natural numbers and $\mathcal{E}$ be the even natural numbers. Given $\mathcal{O} \approx \mathcal{E}$, then $\mathcal{P}(\mathcal{O}) \approx \mathcal{P}(\mathcal{E})$. Since cardinality is defined as a comparison between the elements of two sets, comparing the cardinality of $\mathcal{O}$ to $\mathcal{P}(\mathcal{O})$, should be no different than comparing the cardinality of $\mathcal{O}$ to $\mathcal{P}(\mathcal{E})$.

Applying Cantor's proof to $|\mathcal{O}| \neq |\mathcal{P}(\mathcal{E})|$, suppose there exists $g : \mathcal{O} \xrightarrow[\text{1-1}]{\text{onto}} \mathcal{P}(\mathcal{E})$. Let $B = \{y \in \mathcal{O} : y \notin g(y)\}$. Then, $B = \mathcal{O}$, however, $\mathcal{O} \notin \mathcal{P}(\mathcal{E})$. So, there does not exist a $z \in \mathcal{O}$ such that $g(z) = B$. Hence, the set, B, cannot be used to create a contradiction. Therefore, the proof is invalid. □

In case the implied mapping is not immediately clear, consider the process of modifying Cantor's proof to show that $|\mathcal{O}| \neq |\mathcal{P}(\mathcal{E})|$.

First, let $B = \{y \in \mathcal{O} : y + 1 \notin g(y)\}$. Notice that once again $B \subseteq \mathcal{O}$ and $B \notin \mathcal{P}(\mathcal{E})$. So, for the same reason as before, the proof is invalid.

Now, try letting $B = \{y \in \mathcal{E} : y \notin g(y - 1)\}$, then $B \in \mathcal{P}(\mathcal{E})$. However, $B \nsubseteq \mathcal{O}$, so for any $b \in B$, $g(b)$ is undefined. Once again, the proof is invalid.

Clearly, the set B cannot be both a subset of the domain and an element in co-domain. In order to repeat Cantor's proof with the odd natural numbers and the power set of the even natural numbers, two separate sets must be used. The modified version of (the second part of) Cantor's proof is as follows:

Cantor's Proof (Modified): Applied to $|\mathcal{O}| \approx |\mathcal{P}(\mathcal{E})|$

Suppose $|\mathcal{O}| = |\mathcal{P}(\mathcal{E})|$, which is to say $\mathcal{O} \approx \mathcal{P}(\mathcal{E})$. Then, there exists $g : \mathcal{O} \xrightarrow[\textit{onto}]{1-1} \mathcal{P}(\mathcal{E})$. Let $B = \{y \in \mathcal{O} : y + 1 \notin g(y)\}$ and let $C = \{x + 1 : x \in B\}$. Then, $B \subseteq \mathcal{O}$ and $C \in \mathcal{P}(\mathcal{E})$. Since g is onto $\mathcal{P}(\mathcal{E})$, there exists some $z \in \mathcal{O}$ such that $g(z) = C$. Either $z \in B$ or $z \notin B$. If $z \in B$, then $g(z) \neq C$, since for any $z \in B$, $z + 1 \notin g(z)$, yet $z + 1 \in C$. If $z \notin B$, then $z + 1 \in g(z)$, however, $z + 1 \notin C$ since $z \notin B$, so $g(z) \neq C$. Hence, in all cases, $g(z) \neq C$, which is a contradiction, since g was assumed to be onto $\mathcal{P}(\mathcal{E})$. Therefore, $|\mathcal{O}| \neq |\mathcal{P}(\mathcal{E})|$. □

By removing the shared symbols between the domain and the co-domain, the implied mapping had to be turned into an explicit mapping in order to construct the proof. Although, the mapping wasn't formally introduced, the odd natural numbers are repeatedly mapped to the even natural numbers by adding one to the odd numbers.

In the original proof to Cantor's Theorem, the implied mapping, f, is obscured by the fact that the power set is generated from the elements of A. This mapping can be made explicit by setting $f(a) = a_c$ for all $a \in A$ where a_c has the same symbol as $a \in A$, but is a sub-element of the co-domain, $\mathcal{P}(A_c)$. The co-domain has been marked with a "c" in order better distinguish between the domain and the co-domain (since they are composed of the same symbols). By replacing the implicit mapping with an explicit mapping, the second part of Cantor's original proof can be re-written as follows:

Cantor's Proof (Modified): Explicit mapping from A to A_c

Suppose $|\mathcal{A}| = |\mathcal{P}(A_c)|$, which is to say $\mathcal{A} \approx \mathcal{P}(A_c)$. Then, there exists $g : \mathcal{A} \xrightarrow[onto]{1-1} \mathcal{P}(A_c)$. And, let there be a function, $f : A \xrightarrow[1-1]{} A_c$ where $f(a) = a_c$ for all $a \in A$, such that a_c is the same symbol as a. Also, let $B = \{y \in A : f(y) \notin g(y)\}$ and let $C = \{f(x) : x \in B\}$. Then, $B \subseteq A$ and $C \in \mathcal{P}(A_c)$. Since g is onto $\mathcal{P}(\mathcal{A})$, there exists some $z \in A$ such that $g(z) = C$. Either $z \in B$ or $z \notin B$. If $z \in B$, then $g(z) \neq C$, since for any $z \in B$, $f(z) \notin g(z)$, yet $f(z) \in C$. If $z \notin B$, then $f(z) \in g(z)$, however, $f(z) \notin C$ since $z \notin B$, so $g(z) \neq C$. Hence, in all cases, $g(z) \neq C$, which is a contradiction, since g was assumed to be onto $\mathcal{P}(\mathcal{A})$. Therefore, $|A| \neq |\mathcal{P}(A_c)|$. □

At first glance, the proof to Cantor's Theorem still seems to be perfectly valid. However, in truth, the proof depends on the function, f, being one-to-one in order to create a contradiction. And, by imposing the function, f, between the domain and co-domain, a relationship is formed which restricts the function, g, to a proper subset of all mappings. As a consequence, the arbitrary function, g, is no longer arbitrary.

In order to prove that the function, f, must be one-to-one, assume the opposite; that the function, f, is not one-to-one. In that case, there must exist $z, z_1 \in A$ such that $f(z) = f(z_1)$. And, since g is arbitrary, we can let $f(z) \notin g(z)$ and $f(z_1) \in g(z_1)$. Which means that $z \in B$ and $z_1 \notin B$. Hence, $f(z) \in C$ since $z \in B$, and, consequently, $f(z_1) \in C$ since $f(z_1) = f(z)$. However, the proof claims that when $z_1 \notin B$, then $f(z_1) \notin C$, which is a contradiction since $f(z_1) \in C$. Thus, when $z_1 \notin B$, $g(z_1)$ may actually equal C. Therefore, the proof to Cantor's Theorem is not valid when the function, f, is not one-to-one.

Furthermore, to show that the function, f, is interjected between the domain and the co-domain, consider allowing the function, f, to be a bijection onto the sub-codomain, A_c (instead of only one-to-one). Then, an alternative function, F^{-1} can be defined as $F^{-1}(Z) = \{f^{-1}(z) : z \in Z\}$ for any $Z \subset A_c$ since every element of A_c has an inverse image under f. And, consequently, the set B can be rewritten as $B = \{y \in A : y \notin F^{-1}(g(y))\}$, which shows how the function, g, is effectively superseded by the function, f. However, since the proof only uses the images of f, and not their inverses, it is unnecessary for the function, f, to be onto $\mathcal{P}(A_c)$.

By imposing the function, f, between the domain, A, and co-domain, $\mathcal{P}(A_c)$, every image of the function, g, is expected to contain the image of f (for the same element of A). And, because the function, f, is one-to-one, only a single element in the domain can map to any specific element in A_c. This unnecessarily limits the function, g, to a proper subset of all mappings even though the function, g, is supposed to be arbitrary.

To illustrate the effect on the function, g, allow the set A to be replaced by the natural numbers in the original proof to Cantor's Theorem. Since the original proof expected the elements in the domain to be mapped to the same elements in its powerset, the function, f, can be defined as $f(n) = n$ for all $n \in \mathbb{N}$. Which results in the proof assuming that the function, g, must fall within the following subset of mappings:

$$g \subset \begin{cases} (1,\{1\}) & (2,\{2\}) & (3,\{3\}) & (4,\{4\}) & \dots \\ (1,\{1,2\}) & (2,\{1,2\}) & (3,\{1,3\}) & (4,\{1,4\}) & \dots \\ (1,\{1,3\}) & (2,\{2,3\}) & (3,\{2,3\}) & (4,\{2,4\}) & \dots \\ (1,\{1,2,3\}) & (2,\{1,2,3\}) & (3,\{1,2,3\}) & (4,\{3,4\}) & \dots \\ (1,\{1,4\}) & (2,\{2,4\}) & (3,\{3,4\}) & (4,\{1,2,4\}) & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{cases}$$

Importantly, notice that the subset assumes that the function, g, cannot map an element in the domain to a set in $\mathcal{P}(\mathbb{N}_c)$ without the same element (i.e., $(1,\{2\}) \notin g$ and $(3,\{1,2\}) \notin g$).

The proof's assumption that the function, g, must fall within this subset can be verified in the proof itself, by setting $g = \{(y,\{f(y)\}) : y \in A\}$. This results in B being the null set, since there does not exist a $y \in A$ such that $f(y) \notin g(y)$. Consequently, $C = \varnothing$ since $B = \varnothing$. And, because the null set in A can be mapped to the null set in $\mathcal{P}(A_c)$, the proof does not create a contradiction, even though the function, g, is not onto the power set.[2]

Thus, in order to create a contradiction, the proof not only assumed that the function, g, was limited to a proper subset of mappings, but also that the function, g, had a mapping outside of the assumed relationship. Although this could be viewed as the proof assuming it's own contradiction, it would be more accurate to say that the proof relied on the reader to intuitively know that a function from the assumed subset could not be onto the power set. So, instead, the reader chooses a function, g, which generates a contradiction and concludes the proof is valid. However, the contradiction only occurred because the function, g, was limited to a proper subset of the mappings in the first place. And, by placing this limit on the function, g, it is not actually arbitrary. Therefore, the proof is invalid.

Disproving the proof that $(0,1) \not\approx \mathbb{N}$

At the heart of the flaw in the proof that $(0,1) \not\approx \mathbb{N}$ lies a fundamental issue with the treatment of the natural numbers. Generally, the natural numbers are accepted to be both a set and a sequence at the same time. However, a set is not a sequence and a sequence is not a set. Sequences have an infinite number of steps, whereas sets do not have any steps at all. And, since an object cannot have both an infinite number of steps and zero steps, it is impossible for an object to be both a set and a sequence at the same time. Therefore, the natural numbers cannot be treated as

[2] To prove that if f is one-to-one then g cannot be onto $P(A)$, assume that there are functions, $g : \mathcal{A} \xrightarrow[onto]{1-1} \mathcal{P}(A_c)$ and $f : A \xrightarrow[1-1]{} A_c$. Furthermore, let $g \subset \{(x,X) : x \in A \wedge X \subset \mathcal{P}(A_c) \wedge f(x) \in X\}$. Then, either $(\forall x \in A)(g(x) = \{f(x)\}) \vee (\exists x \in A)(g(x) \neq \{f(x)\})$. If $(\forall x \in A)(g(x) = \{f(x)\})$ then let $B = \{f(x_1), f(x_2)\}$ for any $x_1, x_2 \in A$. Since every image of g only has one element, there does not exist an $x \in A$ such that $g(x) = B$. Hence, g is not onto $\mathcal{P}(A_c)$. Otherwise, if $(\exists x \in A)(g(x) \neq \{f(x)\})$, then let $B = \{f(x_1)\}$ for some $x_1 \in A$ where $g(x_1) \neq \{f(x_1)\}$. Since f is one-to-one and $(\forall x \in A)(f(x) \in g(x))$, there does not exist an $x \in A$ such that $g(x) = B$. Hence, g is still not onto $\mathcal{P}(A_c)$. Therefore, if $g \subset \{(x,X) : x \in A \wedge X \subset \mathcal{P}(A_c) \wedge f(x) \in X\}$ for any $f : A \xrightarrow[1-1]{} A_c$, then g is not onto $\mathcal{P}(A_c)$.

both a set and a sequence simultaneously. [3]

Since sequences have an infinite number of steps, they are often fully defined through the use of a deterministic algorithm, which allows it to be stated whether or not an element exists in the sequence. For example, if a sequence is defined as $S_n = \{\frac{1}{n} : \forall n \in \mathbb{N}\}$, then it can be definitively stated that for any $n \in \mathbb{N}$, $S_n \neq 2$. This statement is rather clear, because a value was defined for every step in the sequence and those values are strictly decreasing from an initial value of 1. However, if only the first three elements of a sequence, T_m, were defined, as say $T_1 = 1$, $T_2 = \frac{1}{2}$, and $T_3 = \frac{1}{3}$, then it could not be said that for all $m \in \mathbb{N}$ that $T_m \neq 2$, since there may be some not-yet-defined step in the sequence which has a value of 2.

Although this seems to be an obvious statement, the proof that $(0,1) \not\approx \mathbb{N}$ essentially makes this claim. Despite characterizing, f, as a function from all natural numbers onto $(0,1)$, the proof actually defines f sequentially to a specific natural number, n. From there, a value, b, is constructed from the images of f, in order to create a value outside of the range. However, the proof neglects to mention that the number, i, which is used to construct the value, b, cannot exceed the number of defined images, n. This leaves the reader with the false impression that the entire sequence has been iterated through, even though only a finite number of images have been defined. And, when the proof takes into account that the sequence has only been partially defined, it immediately becomes apparent that proof does not show that $(0,1)$ is denumerable. After all, the same logic could be used to argue that the cardinality of odd natural numbers is greater than the cardinality of the natural numbers themselves (which is obviously untrue).

Disproof: Proof of $(0,1) \not\approx \mathbb{N}$

Suppose $\mathcal{O}$ is denumerable, where $\mathcal{O}$ are the odd natural numbers. Then there is a function $f : \mathbb{N} \to \mathcal{O}$ that is one-to-one and onto $\mathcal{O}$. Write the images of f, for each $n \in \mathbb{N}$, in normalized form:

$$\begin{aligned}
f(1) &= a_{1m_1} \dots a_{12}a_{11}a_{10} \\
f(2) &= a_{2m_2} \dots a_{22}a_{21}a_{20} \\
f(3) &= a_{3m_3} \dots a_{32}a_{31}a_{30} \\
f(4) &= a_{4m_4} \dots a_{42}a_{41}a_{40} \\
&\vdots \\
f(n) &= a_{nm_n} \dots a_{n2}a_{n1}a_{n0} \\
&\vdots
\end{aligned}$$

Now let b be the number $b = b_n \dots b_2 b_1$.

$$b_i = \begin{cases} 5 & \text{if } a_{ii} \neq 5 \text{ or } a_{ii} \text{ D.N.E.} \\ 3 & \text{if } a_{ii} = 5 \end{cases}$$

Then $b \in \mathcal{O}$ because of the way it has been constructed. However, for each natural number, n, b for $f(n)$ differs in the n'th decimal place. Thus $b \neq f(n)$ for any $n \in \mathbb{N}$,

[3]Furthermore, a sequence is not a set, because sequences are not necessarily equal if they contain the same elements (or, in other words, the Axiom of Extensionality does not apply to sequences).

which means $b \notin Rng(f)$. Thus, f is not onto $\mathcal{O}$. This contradicts our assumption that f is onto $\mathcal{O}$. Therefore, $\mathcal{O}$ is not denumerable. However, $\mathcal{O}$ is denumerable. Therefore, the proof is invalid. □

The odd natural numbers seem to have a larger cardinality than the natural numbers themselves, because the odd natural numbers were represented by a set instead of a sequence. And, since sets do not have any steps, all of the odd natural numbers must be available immediately. Therefore, regardless of the step, there will always be another odd natural number which has not been represented in the sequence of the natural numbers. [4]

An alternative way of viewing the original proof is that the function, f, was fully defined from the beginning and only the value, b, was constructed sequentially. In that case, each step in the construction of the value, b, was chosen so that it differed from the images of the function, f, from one to that step. And, since each step in the sequence was specifically chosen so that it only differed from the previous step by its least significant digit, the sequence converges to a single value, b, in the open interval between zero and one.

On the surface, it might seem as if the value, b, is outside of the range of the sequence due to the way it was constructed. However, simply because a value does not exist in the function up until that point, does not mean the value does not exist in the function at some future point.

Consider applying the proof to the function, $f : \mathbb{N} \to (0,1)$ where $f(x) = 0.x_1x_2x_3\ldots$ and $x \in \mathbb{N}$ is represented as $x = \ldots x_3x_2x_1$. The function, f, can be listed out sequentially as $(f(1), f(2), f(3), \ldots)$ which equals $(0.1, 0.2, 0.3, \ldots)$. Then, the value, b, can be constructed sequentially as $(0.5, 0.55, 0.555, \ldots)$ which approaches the value $0.\overline{5}$. However, notice that every value in the sequence used to construct the value, b, also exists in the sequence of f. Specifically, the subsequence $(f(5), f(55), f(555), \ldots)$ equals $(0.5, 0.55, 0.555, \ldots)$ which approaches the value $0.\overline{5}$ as well. Thus, the value, b, was not constructed outside the range of the sequence of f, but rather, it was constructed at a different rate than it was constructed in the sequence of f.

Therefore, the proof was incorrect when it stated that the value, b, was not an element in the sequence because of the way it was constructed. Instead, it would be more accurate to say that the value, b, was not an element in the sequence at the time it was constructed.

The Natural Numbers as a Set

In the process of discussing the flaw in the proof that $(0,1) \not\approx \mathbb{N}$, the open interval was replaced with the odd natural numbers, in order to demonstrate how the logic of the proof could also be used to claim that the odd natural numbers had a greater cardinality than all of the natural numbers. Although the logic was clearly flawed, it demonstrated that the sequence of the natural numbers only has a finite number of elements at any specific step. And, therefore, the sequence of natural numbers only has an infinite number of elements, because it has an infinite number of steps.

[4]The proof requires the value, n, as an end point to the sequence, because without it, the sequence would continue to be defined forever and the proof could not progress to the next argument (similar to a programmatic infinite loop).

Unlike the sequence of the natural numbers, the set of the natural numbers do not have any steps. Instead each element in the set of the natural numbers is solely represented by a unique symbol. This symbol typically consists of a finite number of digits from a specific base (e.g., the decimal system, binary system, etc). While this representation is perfectly adequate when considering the natural numbers as a sequence, it is inadequate when describing them as a set. Limiting the format of a natural number to a finite number of digits, say n, with a finite base, say b, can only yield b^n natural numbers, which is finite (since both of the components are finite). And, since the sequence of the natural numbers has an infinite number of elements, the set of the natural numbers need to have an infinite number of elements as well.

A simple way to resolve this issue is to allow the elements in the set of the natural numbers to be represented with an infinite number of digits. This allows the set of the natural numbers to be fully defined with an infinite number of elements from the onset, while still maintaining a familiar representation of the natural numbers. Thus, in the following proofs, elements in the set of natural numbers will be represented with an infinite number of digits. Any natural number that can be defined with a finite number of digits will be proceeded by infinite number of zeros.

Theorem 1. $\mathbb{N} \approx (0,1)$

The set of natural numbers has the same cardinality as the open interval, $(0,1)$.

Proof: $\mathbb{N} \approx (0,1)$

Let $f : \mathbb{N} \to (0,1)$ be defined as $f(x) = 0.x_1x_2x_3\ldots$, where $x \in \mathbb{N}$ is represented in decimal format as $x = \ldots x_3x_2x_1$.

Let $f(y) = f(z)$ for some $y, z \in \mathbb{N}$. Then, represent $f(y) = 0.y_1y_2y_3\ldots$ and $f(z) = 0.z_1z_2z_3\ldots$. By the definition of f, $y = \ldots y_3y_2y_1$ and $z = \ldots z_3z_2z_1$. And, since $f(y) = f(z)$, then $y_i = z_i$ for all $i \in \mathbb{N}$. Thus $y = z$ and, therefore, f is one-to-one.

Choose any $p \in (0,1)$ where $p = 0.p_1p_2p_3\ldots$. Notice if $u = \ldots p_3p_2p_1$, then $f(u) = 0.p_1p_2p_3\ldots = p$. Hence, f is onto $(0,1)$.

Since f is one-to-one and onto $(0,1)$, then f is a bijection. Therefore, $\mathbb{N} \approx (0,1)$. □

Corollary 1. $\mathbb{N} \approx \mathfrak{R}$

The set of natural numbers has the same cardinality as the real numbers.

Proof: $\mathbb{N} \approx \mathfrak{R}$

Given $\mathbb{N} \approx (0,1)$ and $(0,1) \approx \mathfrak{R}$, then $\mathbb{N} \approx \mathfrak{R}$. [5] □

Theorem 2. $\mathbb{N} \approx \mathcal{P}(\mathbb{N})$

The set of natural numbers has the same cardinality as the power set of the natural numbers.

Proof: $\mathbb{N} \approx \mathcal{P}(\mathbb{N})$

Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be defined for all $x \in \mathbb{N}$ where x is represented in binary as

[5] If the proof that $\mathbb{N} \approx \mathfrak{R}$ does not seem satisfying, then consider the bijection, $f : \mathbb{N} \to \mathfrak{R}$, where the natural numbers are represented in binary. Define f such that for any $n \in \mathbb{N}$, if $n \bmod 2 = 0$ then $f(n)$ is negative, otherwise it is positive. The digits of f(n) left of the decimal point are the same as n where the powers of two are odd. And, the digits of $f(n)$ right of the decimal point are the same as n where the powers of two are even except mirrored around the decimal place (excluding 2^0). For example, $f(11000) = -10.01$ and $f(100010001) = 0000.0101 = 0.0101$.

$x = \ldots x_3 x_2 x_1$ and $f(x) = \{i \in \mathbb{N} : x_i = 1\}$.

For the sake of contradiction, assume there exists $x, y \in \mathbb{N}$ such that $f(x) = f(y)$ and $x \neq y$. For any $i \in \mathbb{N}$, if $i \in f(x)$ then $x_i = 1$ and since $f(x) = f(y)$, then $i \in f(y)$ and $y_i = 1$. Similarly, if $i \notin f(x)$ then $x_i = 0$ and since $f(x) = f(y)$, then $i \notin f(y)$ and $y_i = 0$. Hence, $x = y$ since $x_i = y_i$ for all $i \in \mathbb{N}$, which is a contradiction since it was assumed that $x \neq y$. Thus, whenever $f(x) = f(y)$, $x = y$. Therefore, f is one-to-one.

For any $Z \in \mathcal{P}(\mathbb{N})$, then Z can be represented as $\{z_1, z_2, \ldots\}$. Let $z = \sum_{j=1}^{\infty} (2^{(z_j - 1)})$ and notice that $f(z) = Z$. Hence, for all $Z \in \mathcal{P}(\mathbb{N})$, there exists an element in the natural numbers, namely z, such that $f(z) = Z$. Therefore, f is onto $\mathcal{P}(\mathbb{N})$. [6]

Since, there exists a one-to-one function, f, from $\mathbb{N}$ onto $\mathcal{P}(\mathbb{N})$, then $\mathcal{P}(\mathbb{N})$ is denumerable. Therefore, $\mathcal{P}(\mathbb{N}) \approx \mathbb{N}$.

□

The Natural Numbers as a Sequence

Proving that the set of natural numbers is equivalent to the real numbers and the power set of the natural numbers is rather straight-forward, since the concept of cardinality is defined between two sets. However, the natural numbers have traditionally been defined as a sequence through Peano's Axioms. This process adds a new natural number with each step through an infinite number of steps (commonly referred to as counting). Although Peano's Axioms define the natural numbers as the process of counting, it is more accurate to say that the natural numbers are the union of all of the elements defined by counting. Or, in other words, counting is a sequence, while the natural numbers are its series (using union in place of addition).

The same underlying concept to bijections can be used to show that a sequence is equivalent to a set. The one-to-one aspect of cardinality is rather trivial when discussing a sequence, since any elements that are duplicated can be removed as long as new elements are continually being added to the sequence. However, for simplicity, the proofs in this section will only map the sequences to unique elements. The concept of a sequence being onto a set is somewhat more complicated. Clearly, an element of the set is mapped to by a sequence, if there exists a step in the sequence that corresponds to the element. However, since there are an infinite number of steps in a sequence, some elements will never be reached. Therefore, an element will also be considered a member of the sequence, if there exists a convergent subsequence that approaches the element. [7]

Of course, in order to have the concept of convergence, the set must also have a metric. In the case of mapping a sequence to the $(0,1)$, the usual metric will be sufficient. Furthermore, to distinguish between the set of natural numbers and the sequence of the natural numbers, the sequence of natural numbers will be symbolized as $\mathbb{N}_q$. Hence, the proof that the sequence of the natural numbers is equivalent to the $(0,1)$ with the usual metric, will be represented as $\mathbb{N}_q \approx ((0,1), ||)$. Finally, since any step in a sequence is preceded by a determinable number of prior steps,

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[6] Since every set must contain the null set, the null set in $\mathbb{N}$ can be mapped to the null set in $\mathcal{P}(\mathbb{N})$. If this doesn't seem sufficient, then one can be subtracted from x and $f(0)$ can be mapped to the null set.

[7] Or, in other words, a value is considered a member of the sequence, if the value is approached by the sequence, not just if the sequence approaches the value.

each step will be represented with a finite number of digits.

Theorem 3. $\mathbb{N}_q \approx ((0,1),||)$

The sequence of the natural numbers has the same cardinality as the open interval, $(0,1)$*, with the usual metric.*

Proof: $\mathbb{N}_q \approx ((0,1),||)$

Define the sequence, $S_i = .i_1 i_2 \dots i_j$ where any $i \in \mathbb{N}_q$ is represent by the decimal expansion $i = i_j i_{j-1} \dots i_1$. Notice that since S is a sequence, each step, i, can be represented with a finite number of digits.

Assume for the sake of contradiction, that there exists $a, b \in \mathbb{N}_q$ such that $S_a = S_b$ and $a \neq b$. Let $S_a = .a_1 a_2 \dots a_c$ and $S_b = .b_1 b_2 \dots b_d$, where $a = a_c \dots a_1$ and $b = b_d \dots b_1$. Since $S_a = S_b$, then $.a_1 a_2 \dots a_c = .b_1 b_2 \dots b_d$. Without loss of generality, let $d > c$. Then, $b_{c+1}, \dots, b_d = 0$ since $S_a = S_b$, and thus, S_b can be rewritten as $.b_1 b_2 \dots b_c$. And, since $S_a = S_b$, $a_e = b_e$ for all $e \leq c$. Hence $a = a_c \dots a_2 a_1 = b_c \dots b_2 b_1 = b$, which is a contradiction since $a \neq b$. Thus, whenever $S_a = S_b$, $a = b$. Therefore, every element in the sequence, S, is distinct (i.e., S is one-to-one).

Choose any element, $x \in (0,1)$. If x can be represented by a finite number of digits, then x can be represented in decimal format as $x = .x_1 x_2 x_3 \dots x_{n-1} x_n$. Then, let $y = x_n x_{n-1} \dots x_1$, and notice that $S_y = x$ by the way the sequence was constructed.

Otherwise, if x cannot be written with a finite number of digits, let x be represented $x = .x_1 x_2 x_3 \dots$ and choose the initial point for the subsequence, K, from S, such that $K_1 = S_{x_u \dots x_1} = .x_1 \dots x_u$ where x_u is the first non-zero value in x and u is its position. For every subsequent step after $m \in \mathbb{N}_q$, find the next non-zero element in x, signified x_w, and let $K_{m+1} = S_{x_w \dots x_v \dots x_1} = .x_1 \dots x_v \dots x_w$ where x_v is the previous non-zero value in x. Notice that, for all $m \in \mathbb{N}_q$, $K_m, K_{m+1} \in S$ and K_{m+1} is after K_m in S, since $x_w \dots x_v \dots x_1 > x_v \dots x_1$. Hence, K is a subsequence of S.

Choose any $\epsilon \in \mathfrak{R}$ such that $\epsilon > 0$. If $\epsilon \geq 1$, then choose $p = 1$. Since, for any $p \in \mathbb{N}_q$, $|x - K_p| < 1$ (since $0 < x, K_p < 1$). Otherwise, if $0 < \epsilon < 1$, then represent ϵ as the decimal expansion $\epsilon = \epsilon_1 \epsilon_2 \epsilon_3 \dots$. Let p be the location of the most significant digit in ϵ. Then, notice $|x - K_p| < \epsilon$, since the p'th step in the subsequence differs from x at the $(p+1)$'th digit (if not further). And, since for all $\epsilon > 0$ there exists a $p \in \mathbb{N}_q$, such that $|x - K_p| \leq \epsilon$, then K converges to x.

Thus, for any $x \in (0,1)$, either there exists either a step, $y \in \mathbb{N}_q$, such that $S_y = x$, or there exists a subsequence, $K \subset S$, such that K converges to x. Therefore, the sequence, S, is onto $(0,1)$ with the usual metric.

Therefore, since the sequence S is one-to-one and onto $(0,1)$ with the usual metric, $\mathbb{N}_q \approx ((0,1),||)$. □

Corollary 2. $\mathbb{N}_q \approx (\mathfrak{R},||)$

The sequence of the natural numbers has the same cardinality as the real numbers with the usual metric.

Proof: $\mathbb{N}_q \approx (\mathfrak{R},||)$

Given $\mathbb{N}_q \approx ((0,1),||)$ and $(0,1) \approx \mathfrak{R}$, then $\mathbb{N}_q \approx (\mathfrak{R},||)$. □

Intuitively, the sequence of the natural numbers should be considered equivalent to the set of the natural numbers when every element in the sequence is mapped to the same element in the set. Unfortunately, it is impossible to converge to any element in the set with an infinite number of digits using the usual metric, since the difference between any two non-equal natural numbers will always be at least one. This means that when ϵ is less than one, there can never be a step in the sequence with a difference less than ϵ. And, therefore, by the standard definition of convergence, these elements can never reached by the sequence.

Yet, the sequence of the natural numbers is clearly equivalent to the set of the natural numbers. So, in order to resolve the issue, a sequence will be said to approach an element in the set of natural numbers, if the distance to that element is strictly decreasing at a steady or accelerating rate. And, since every metric is bounded below by zero, the distance must go to zero as the sequence goes to infinity. Therefore, the sequence approaches the element.

At this point, it is possible to show that the set and sequence of the natural numbers are equivalent. Every element in the set is the natural numbers with a finite number of digits can be mapped directly to the same element in the sequence of natural numbers. And, every element with an infinite number of digits has a subsequence in the sequence of the natural numbers which approaches its value using the usual metric. Specifically, any element in the set with an infinite number of digits, n, can be represented as $n = \ldots n_3 n_2 n_1$. Which allows the subsequence $\{n_{i_1} \ldots n_1,\ n_{i_2} \ldots n_{i_1} \ldots n_1,\ \ldots\}$ to be chosen from $\mathbb{N}_q$, where $n_{i_{x+1}}$ is the next non-zero number in n after n_{i_x} for all $x \in \mathbb{N}_q$. Notice that the subsequence approaches the element, n, at an accelerating rate under the usual metric. Hence, any element in the set of the natural numbers with an infinite number of digits has a subsequence in the sequence of the natural numbers that approaches the element. And, therefore, the sequence of the natural numbers is equivalent to the set of the natural numbers.

The same methodology can be applied to the sequence of the natural numbers to show that it is equivalent to the power set of the natural numbers. Of course, the usual metric cannot be applied to a set, so instead a different metric will have to be used.

Theorem 4. $\mathbb{N}_q \approx (\mathcal{P}(\mathbb{N}), \mathcal{C})$

The sequence of the natural numbers has the same cardinality as the power set of the natural numbers when the metric is the number of differing elements.

Lemma 1. *The number of differing elements, $\mathcal{C}$, is a metric on $\mathcal{P}(\mathbb{N})$*

Proof. $\mathcal{C}$ is a metric on $\mathcal{P}(\mathbb{N})$

Define $\mathcal{C} : S_1 X S_2 \to \mathbb{N}$ as the number of elements which differ between the two sets, $S_1, S_2 \in \mathcal{P}(\mathbb{N})$. Or, more formally, $\mathcal{C}(S_1, S_2) = |\{x \in S_1 : x \notin S_2\}| + |\{y \in S_2 : y \notin S_1\}|$ where $||$ is the cardinality of the sets.

(i) *Positivity.* $\mathcal{C}(S_1, S_2) \geq 0$ for all $S_1, S_2 \in \mathcal{P}(\mathbb{N})$, since $|\{x \in S_1 : x \notin S_2\}| \geq 0$ and $|\{y \in S_2 : y \notin S_1\}| \geq 0$. Additionally, $\mathcal{C}(S_1, S_2) = 0$ iff $|\{x \in S_1 : x \notin S_2\}| = 0$ and $|\{y \in S_2 : y \notin S_1\}| = 0$ iff $(\forall x \in S_1) x \in S_2$ and $(\forall y \in S_2) y \in S_1$ iff $S_1 = S_2$.

(ii) *Symmetry.* For all $S_1, S_2 \in \mathcal{P}(\mathbb{N})$, $\mathcal{C}(S_1, S_2) = |\{x \in S_1 : x \notin S_2\}| + |\{y \in S_2 : y \notin S_1\}| = |\{y \in S_2 : y \notin S_1\}| + |\{x \in S_1 : x \notin S_2\}| = \mathcal{C}(S_2, S_1)$.

(iii) *The Triangle Inequality.* The following will show that for any $S_1, S_2, S_3 \in \mathcal{P}(\mathbb{N})$, $\mathcal{C}(S_1, S_3) \leq \mathcal{C}(S_1, S_2) + \mathcal{C}(S_2, S_3)$:

$\mathcal{C}(S_1, S_2) + \mathcal{C}(S_2, S_3) = (|S_2 - S_1| + |S_1 - S_2|) + (|S_3 - S_2| + |S_2 - S_3|)$

$$= (|((S_2 - S_1) - S_3)| + |((S_2 - S_1) \cap S_3)|) + (|((S_1 - S_2) - S_3)| + |((S_1 - S_2) \cap S_3)|)$$
$$+ (|((S_3 - S_2) - S_1)| + |((S_3 - S_2) \cap S_1)|) + (|((S_2 - S_3) - S_1)| + |((S_2 - S_3) \cap S_1)|)$$
$$= (|((S_1 - S_2) - S_3)| + |((S_2 - S_3) \cap S_1)|) + (|((S_3 - S_2) - S_1)| + |((S_2 - S_1) \cap S_3)|)$$
$$+ |((S_2 - S_1) - S_3)| + |((S_2 - S_3) - S_1)| + |((S_1 - S_2) \cap S_3)| + |((S_3 - S_2) \cap S_1)|$$
$$= |S_1 - S_3| + |S_3 - S_1| + 2|((S_2 - S_1) - S_3)| + 2|((S_1 - S_2) \cap S_3)|$$
$$= \mathcal{C}(S_1, S_3) + 2|((S_2 - S_1) - S_3)| + 2|((S_1 - S_2) \cap S_3)| \geq \mathcal{C}(S_1, S_3),$$
since $2|((S_2 - S_1) - S_3)| + 2|((S_1 - S_2) \cap S_3)| \geq 0$. □

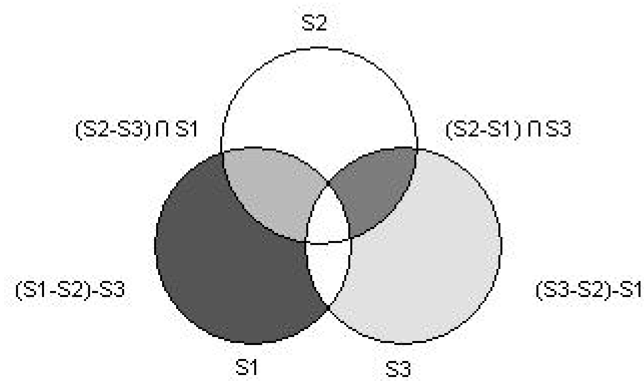

FIGURE 1. $(S_1 - S_3) \cup (S_3 - S_1)$

Proof. $\mathbb{N}_q \approx (\mathcal{P}(\mathbb{N}), \mathcal{C})$

Define the sequence, $S_i = \{k \in \mathbb{N}_q : (i-1)_k = 1\}$ where $i \in \mathbb{N}_q$ is represented in binary as $i = i_1 i_2 \dots i_j$ for some $j \in \mathbb{N}_q$.

By construction, the sequence is one-to-one, since the elements in the sequence are based on the representation of the natural number in binary, which must be unique.

For any $X \in \mathcal{P}(\mathbb{N})$, either X has a finite number elements or X has an infinite number of elements. If X has a finite number of elements, say n, then let $b = (\sum_{c=1}^{n} (2^{(x_c - 1)})) + 1$ where x_c is the c'th element of X. Then, $S_b = X$, so $X \in S$.

If X has an infinite number of elements, consider the subsequence, $K \subset S$, defined by $K_m = S_y$ where $y = (\sum_{p=1}^{m} (2^{(x_p - 1)})) + 1$, m is the m'th element in the subsequence, and x_p is the p'th element in X. Then, for any $z \in \mathbb{N}_q$, $\mathcal{C}(K_{z+1}, X) = \mathcal{C}(K_z, X) - 1$ or, in other words, K is strictly decreasing by a rate of one per step. Also, notice that $\mathcal{C}$ is bounded below by zero, since $\mathcal{C}$ is a metric. Hence, $\mathcal{C}(K_z, X) \to 0$ as $z \to \infty$. Therefore, as K goes to infinity, K approaches X.

For any $X \in \mathcal{P}(\mathbb{N})$, either there exists a $b \in \mathbb{N}_q$ such that $S_b = X$, or there exists a subsequence, $K \subset S$, such that as K goes to infinity, K approaches X. Thus, S is onto $(\mathcal{P}(\mathbb{N}), \mathcal{C})$.

Hence, S is one-to-one and onto $(\mathcal{P}(\mathbb{N}), \mathcal{C})$. Therefore, $\mathbb{N}_q \approx (\mathcal{P}(\mathbb{N}), \mathcal{C})$.[8]

□

Conclusion

The paper began by discussing the flaws in two proofs, which were key to showing that the natural numbers are not equivalent to the real numbers and the power set of the natural numbers, namely Cantor's Theorem and the proof that $(0,1) \not\approx \mathbb{N}$. Both of these proofs were flawed in their own way. The proof to Cantor's Theorem unintentionally assumed that the elements in the domain must be mapped to the same elements in the co-domain. And, consequently, the arbitrary function in proof was restricted to a proper subset of possible mappings, which made the proof invalid. In the case of the proof that $(0,1) \not\approx \mathbb{N}$, the proof sequentially constructed a value which was meant to be outside of the range of an arbitrary function. However, by modifying the value at each step in the sequence, the proof only created a value that was outside of the sequence up to that particular step. Which meant that the value could still exist at some later point in the sequence. And, therefore, the proof that $(0,1) \not\approx \mathbb{N}$ was invalid as well.

The flaw in the proof that $(0,1) \not\approx \mathbb{N}$ helped demonstrate a key difference between a sequence and an infinite set. Sequences have an infinite number of elements because they have an infinite number of steps whereas infinite sets must have all of their elements defined from the onset. The natural numbers have commonly been considered both a sequence and a set simultaneously. However, since sequences have a infinite number of steps and sets do not have any steps at all, the natural numbers can not be both at the same time. When the natural numbers are represented as a set, they must have an infinite number of elements defined immediately. Thus, an element in the set of the natural numbers can be represented with an infinite number of digits. On the other hand, when the natural numbers are defined sequentially, it will take an infinite number of steps to reach some of the elements in the sequence. Therefore, an element is considered to be in a sequence if it exists at a specific step or if a subsequence approaches the element.

Intuitively, the sequence of the natural numbers should be equivalent to the set of the natural numbers when every element in the sequence is mapped to the same element in the set. However, because the difference between any two non-equal natural numbers is always greater than or equal to one, the sequence cannot be shown to converge to an element in the set with an infinite number of digits (with the traditional definition of convergence). Instead, a sequence will be said to approach the element in the set if the distance goes to zero at a steady of accelerating rate. This guarantees the sequence can overcome the infinite distance to the element. Which allows the sequence of the natural numbers to be equivalent to the set of the natural numbers as expected.

Once the distinction between the natural numbers as a set and as a sequence has been made, it is almost trivial to prove that the natural numbers are equivalent to real numbers and the power set of the natural numbers. The natural numbers can be mirrored around the decimal point to create the open interval between zero and one, which is equivalent to the real numbers. And, every element in the power

[8] The proof was only written as if the elements were ordered to make it more readable. Any element can be chosen from the set, X, in order to determine the next step in the subsequence as long as that element has not already been selected.

set can be created by using the digits in the binary representation of the natural numbers as switches to determine whether or not a number appears in an element of the power set. Thus, proving that the natural numbers are equivalent to the real numbers and the power set of the natural numbers.

The conclusion that the real numbers and the power set of the natural natural numbers are equivalent to the natural numbers has important ramifications for the concept of cardinality. One of the most direct consequences of these proofs is the irrelevance of the Continuum Hypothesis. Since the natural numbers have been shown to have the same cardinality as the real numbers, there can not be a cardinal number between $\aleph_0$ and $\mathbf{c}$, which means the Continuum Hypothesis is trivially true. More importantly, by showing that the set of the natural numbers is equivalent to the power set of the natural numbers, Cantor's Theorem must not be true. And, without Cantor's Theorem, there is no longer any basis to believe that infinite sets can have different cardinalities.

To the contrary, the Well-Ordering Theorem essentially states that all infinite sets have the same cardinality. By this theorem, every set, including the real numbers and the power set of the natural numbers, has a function which can identify a minimum element and order the elements linearly. Which means, that every infinite set can be mapped to a sequence. And, since the natural numbers can be defined as a sequence, the Well-Ordering Theorem has already shown that every infinite set can be mapped to the natural numbers. Therefore, all infinite sets have the same cardinality as the natural numbers.

Appendix

Note that for all of the proofs below, the cardinality of a set was notated with a double overline instead of ||, since that was the notation used in the source.

Theorem 5 (Cantor's Theorem)**.** *For every set A,* $\overline{\overline{A}} < \overline{\overline{P(A)}}$ [9]

Proof. To show $\overline{\overline{A}} < \overline{\overline{P(A)}}$, we must first show that (i) $\overline{\overline{A}} \leq \overline{\overline{P(A)}}$ and (ii) $\overline{\overline{A}} \neq \overline{\overline{P(A)}}$. Part (i) follows from the fact that $F : A \to P(A)$ defined by $F(x) = \{x\}$ is one-to-one.

To prove (ii), suppose $\overline{\overline{A}} = \overline{\overline{P(A)}}$; that is, assume $A \approx P(A)$. Then there exists $g : A \xrightarrow[onto]{1-1} P(A)$. Let $B = \{y \in A : y \notin g(y)\}$. Since $B \subseteq A$, $B \in P(A)$, and since g is onto $P(A)$, $B = g(z)$ for some $z \in A$. Now either $z \in B$ or $z \notin B$. If $z \in B$, then $z \notin g(z) = B$, a contradiction. Similarly, $z \notin B$ implies $z \notin g(z)$, which implies $z \in B$ again a contradiction. We conclude that A is not equivalent to $P(A)$ and hence $\overline{\overline{A}} < \overline{\overline{P(A)}}$. □

Theorem 6. $(0,1)$ *is Uncountable* [10]

Proof. The interval (0,1) includes the subset $\left\{\frac{1}{2^k} : k \in \mathbb{N}\right\}$, which is infinite. Thus, since every subset of a finite set is finite, (0,1) is infinite.

Suppose (0,1) is denumerable. Then there is a function $f : \mathbb{N} \to (0,1)$ that is one-to-one and onto (0,1). Write the images of f, for each $n \in \mathbb{N}$, in normalized form:

$$\begin{aligned}
f(1) &= 0.a_{11}a_{12}a_{13}a_{14}a_{15}\ldots \\
f(2) &= 0.a_{21}a_{22}a_{23}a_{24}a_{25}\ldots \\
f(3) &= 0.a_{31}a_{32}a_{33}a_{34}a_{35}\ldots \\
f(4) &= 0.a_{41}a_{42}a_{43}a_{44}a_{45}\ldots \\
&\vdots \\
f(n) &= 0.a_{n1}a_{n2}a_{n3}a_{n4}a_{n5}\ldots \\
&\vdots
\end{aligned}$$

Now let b be the number $b = 0.b_1b_2b_3b_4b_5\ldots$, where

$$b_i = \begin{cases} 5 & \text{if } a_{ii} \neq 5 \\ 3 & \text{if } a_{ii} = 5 \end{cases}$$

Then $b \in (0,1)$ because of the way it has been constructed. However, for each natural number n, b differs from $f(n)$ in the nth decimal place. Thus, $b \neq f(n)$ for any $n \in \mathbb{N}$, which means $b \notin Rng(f)$. Thus, f is not onto (0,1). This contradicts our assumption that f is onto (0,1). Therefore, (0,1) is not denumerable. □

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[9] Douglas Smith, Maurice Eggen, and Richard St. Andre, *A Transition to Advanced Mathematics, 7th Edition* (Boston: Brooks/Cole, 2011), p261

[10] Douglas Smith, Maurice Eggen, and Richard St. Andre, *A Transition to Advanced Mathematics, 7th Edition* (Boston: Brooks/Cole, 2011), p246.

Theorem 7. $\Re$ *is Uncountable* [11]

Proof. Define $f : (0,1) \to \Re$ by $f(x) = tan(\pi x - \frac{\pi}{2})$. The function f is a contradiction and translation of one branch of the tangent function and is one-to-one and onto $\Re$. Thus $(0,1) \approx \Re$. □

Theorem 8. $\overline{\overline{P(\mathbb{N})}} = c$ [12]

Proof. First, recall that any real number in the interval $(0,1)$ may be expressed in a base 2 (binary) expansion $0.b_1b_2b_3b_4\ldots$, where each b_i is either 0 or 1. If we exclude sequences that terminate with infinitely many 1's, such as 0.010111111... (which has the same value as 0.0110000...), then the representation is unique. Thus we may define a function $f : (0,1) \to P(\mathbb{N})$ such that for each $x \in (0,1)$,

$$f(x) = \{n \in \mathbb{N} : b_n = 1 \text{ in the binary representation of } x\}$$

The uniqueness of binary representations ensures that the function is defined and is one-to-one. Since f is one-to-one, $\overline{\overline{(0,1)}} \leq \overline{\overline{P(\mathbb{N})}}$.

Next, define $g : P(\mathbb{N}) \to (0,1)$ by $g(A) = 0.a_1a_2a_3a_4\ldots$, where

$$a_n = \begin{cases} 2 & \text{if } n \in A \\ 5 & \text{if } n \notin A \end{cases}$$

For any set $A \subseteq \mathbb{N}$, $g(A)$ is a real number in $(0,1)$ with decimal expansion consisting of 2's and 5's. (Any pair of digits not including 9 will do.) The function g is one-to-one but certainly not onto $(0,1)$. Therefore, $\overline{\overline{P(\mathbb{N})}} \leq \overline{\overline{(0,1)}}$.

By the Cantor-Schröder-Bernstein Theorem, $\overline{\overline{P(\mathbb{N})}} = \overline{\overline{(0,1)}}$. Therefore $\overline{\overline{P(\mathbb{N})}} = \mathbf{c}$. □

References

[SEA] Douglas Smith, Maurice Eggen, and Richard St. Andre, (2011). *A Transition to Advanced Mathematics, 7th Edition.* Brooks/Cole, Boston.

[11] Douglas Smith, Maurice Eggen, and Richard St. Andre, *A Transition to Advanced Mathematics, 7th Edition* (Boston: Brooks/Cole, 2011), p247

[12] Douglas Smith, Maurice Eggen, and Richard St. Andre, *A Transition to Advanced Mathematics, 7th Edition* (Boston: Brooks/Cole, 2011), p265.